PMM U.S.S.R.,Vol.44,pp.622-628
0021-8928/81/5 0622 $7.50/0

UDC 539.3:534.1

ASYMPTOTIC THEORY OF ELASTIC PLATE BUCKLING UNDER LATERAL COMPRESSION*

A. N. RUDEV

Homogeneous solutions obtained in /1/ are used for investigating in three-dimensional formulation the stability of a thick plate of arbitrary shape of neo-Hookean material and free of constraints, /2/. Boundary conditions at lateral surfaces are obtained using the variational principle of superposition of a small deformation on a finite one, as proposed in /3/. As the result, the problem of critical pressure determination is reduced to the general problem of eigenvalues for an infinite homogeneous system of operator equations whose dependence (explicit as well as implicit) on the initial deformation parameters is essentially nonlinear. The asymptotic method proposed in /4/ is extended so as to make possible the determination of critical load asymptotics, as the plate thickness ε approaches zero. The effect of potential solutions that correspond to irregular (with the initial deformation eliminated) roots of the characteristic equation /1/, and have no analogs in the linear theory of elasticity /5/, on the plate stress-strain state and on the magnitude of critical pressure is determined. It is shown that the two-dimensional theory of plate buckling based on the Kirchhoff hypotheses /6/ makes possible the correct determination of the principal terms of the critical load asymptotics as $\varepsilon \to 0$. As an example, the axisymmetric buckling of a circular plate is considered. Five terms of the asymptotic expansion of the critical load are obtained. It is established that the classic theory yields a deficient critical force, with a relative error of the order of ε^2.

1. Let us consider a plate of incompressible neo-Hookean material of thickness $2h$ subjected to uniform pressure t at its side surface. It is assumed that the plate is unrestrained, its end-faces free of stress, that mass forces are absent, and the surface loading is dead. Under these conditions the plate deformation is defined as follows:

$$y_1 = \lambda x_1, \quad y_2 = \lambda x_2, \quad y_3 = \lambda^{-2} x_3 \quad (\lambda = \mathrm{const}) \tag{1.1}$$

where x_k, y_k $(k = 1, 2, 3)$ are Cartesian coordinates, respectively, prior to and after deformation. The relation between t and λ is such that

$$t = G(\lambda^{-4} - \lambda^2) \tag{1.2}$$

where G is the shear modulus of the material. A small bending deformation, defined by the equations of neutral equilibrium in /7,1/, is superposed on the finite deformation (1.1).

We introduce a local system of dimensionless coordinates n, s, ζ /5/. The following notation is used below: τ is the volume of the undeformed plate; o is the plate surface; Γ is the contour bounding the middle plane (assumed reasonably smooth); R is the radius of curvature the contour Γ; $2a$ is a characteristic longitudinal dimension of the plate; $\rho = a / R$ is the dimensionless curvature; $\varepsilon = h / a$ is the plate relative thickness; u, v, w, and ∂_{km} $(k, m = 1, 2, 3)$ are vectors of additional displacement and of the additional Piola stress tensor /2,7/ in the system n, s, ζ with indices 1,2,3 corresponding, respectively, to axes n, s, ζ; p is the indeterminate function of coordinates related to the incompressibility of the material /2/; $H = (1 + \rho n)^{-1}$ is a quantity inverse to the Lamé coefficient of the considered curvilinear coordinate system; ∂_i $(i = 1, 2)$ are operators of differentiation with respect to n and s, respectively, and Δ is the two-dimensional Laplace operator in variables n and s.

In the local coordinate system the homogeneous solutions of neutral equilibrium equations /1/ are of the form $(\gamma \equiv \lambda^{-3}, \ \omega \equiv 1 + \gamma^2)$:

---

*Prikl.Matem.Mekhan.,44,No.5,882-891,1980

The penetrating solution

$$U = a\varepsilon\gamma\zeta\partial_1\Psi + a\varepsilon^3 A(\zeta)\partial_1\Delta\Psi$$
$$V = a\varepsilon\gamma\zeta H\partial_2\Psi + a\varepsilon^3 A(\zeta) H\partial_2\Delta\Psi$$
$$W = -a\Psi + a\varepsilon^2 B(\zeta)\Delta\Psi, \quad P = \varepsilon\lambda C(\zeta)\Delta\Psi$$
$$A(\zeta) = -\gamma\alpha_0^{-2}\zeta + g\alpha_0^{-1}[\omega X_0(1, \gamma\zeta) - 2\gamma X_0(\gamma, \zeta)]$$
$$B(\zeta) = \alpha_0^{-2} + g[\omega Y_0(1, \gamma\zeta) - 2\gamma^2 Y_0(\gamma, \zeta)]$$
$$C(\zeta) = \alpha_0^{-1}\omega\cos\alpha_0 X_0(1, \gamma\zeta), \quad g = \alpha_0^{-2}(\gamma^2 - 1)^{-1}\cos\alpha_0$$
$$X_0(x, y) = \cos\alpha_0 x\sin\alpha_0 y, \quad Y_0(x, y) = \cos\alpha_0 x\cos\alpha_0 y$$
$$\varepsilon^2\Delta^2\Psi - \alpha_0^2\Delta\Psi = 0 \tag{1.3}$$

where α_0 is any of the two pure imaginary roots of the characteristic equation /1/, the vortex solution

$$u^\circ = -a\varepsilon^2\sum_{m=1}^{\infty} F_m(\zeta) H\partial_2 B_m, \quad v^\circ = a\varepsilon^2\sum_{m=1}^{\infty} F_m(\zeta)\partial_1 B_m, \quad w^\circ = p^\circ = 0$$
$$F_m(\zeta) = 4\gamma^2(-1)^{m+1}(1-\gamma^2)^{-1}\sigma_m^{-2}\cos\sigma_m\gamma\sin\sigma_m\zeta$$
$$\varepsilon^2\Delta B_m - \sigma_m^2 B_m = 0, \quad \sigma_m = \pi(2m-1)/2, \quad m = 1, 2, 3, \ldots$$

and the potential solution

$$u_* = a\varepsilon\sum_{q=1}^{\infty} T_q(\zeta)\partial_1 C_q + a\varepsilon\sum_{\nu=1}^{\infty} T_\nu(\zeta)\partial_1 C_\nu$$
$$v_* = a\varepsilon\sum_{q=1}^{\infty} T_q(\zeta) H\partial_2 C_q + a\varepsilon\sum_{\nu=1}^{\infty} T_\nu(\zeta) H\partial_2 C_\nu$$
$$w_* = -a\sum_{q=1}^{\infty} M_q(\zeta) C_q - a\varepsilon\sum_{\nu=1}^{\infty} M_\nu(\zeta) C_\nu$$
$$p_* = \varepsilon^{-1}\lambda\sum_{q=1}^{\infty} N_q(\zeta) C_q + \varepsilon^{-1}\lambda\sum_{\nu=1}^{\infty} N_\nu(\zeta) C_\nu$$
$$T_q(\zeta) = \alpha_q^{-1}\beta[\omega X_q(1, \gamma\zeta) - 2\gamma X_q(\gamma, \zeta)]$$
$$M_q(\zeta) = -\beta[\omega Y_q(1, \gamma\zeta) - 2\gamma^2 Y_q(\gamma, \zeta)]$$
$$N_q(\zeta) = \alpha_q\omega X_q(1, \gamma\zeta), \quad \beta = (\gamma^2 - 1)^{-1}$$
$$X_q(x, y) = \cos\alpha_q x\sin\alpha_q y, \quad Y_q(x, y) = \cos\alpha_q x\cos\alpha_q y$$
$$\varepsilon^2\Delta C_q - \alpha_q^2 C_q = 0, \quad \varepsilon^2\Delta C_\nu - \alpha_\nu^2 C_\nu = 0$$

where α_q and α_ν are, respectively, the complex and real roots of the characteristic equation /1/ that lie in the right-hand half plane.

Subsequently we use also subscripts p and μ for denoting, respectively, the complex and real roots (and related quantities). Functions X_ν and Y_ν are obtained from X_q and Y_q by the substitution in the latter of ν for q, and the expressions for T_ν, M_ν, and N_ν are obtained by a similar substitution in T_q, M_q, and N_q, followed by multiplication by $1/\cos\alpha_\nu$.

We seek the critical value of parameter t (or γ) for which the equations of neutral equilibrium with similar boundary conditions at the plate end-faces

$$\partial_{31}|_{\zeta=\pm 1} = \partial_{32}|_{\zeta=\pm 1} = \partial_{33}|_{\zeta=\pm 1} = 0 \tag{1.4}$$

and at the side face

$$\partial_{11}|_{n=0} = \partial_{12}|_{n=0} = \partial_{13}|_{n=0} = 0 \tag{1.5}$$

which imply absence of additional loading, admit nontrivial solutions. According to /3/ this problem is equivalent to that of stationarity of the functional of the additional potential energy of deformation

$$\Pi = \iiint_\tau E\,d\tau \tag{1.6}$$

where E is the volume density of the additional deformation energy in the metric of the undeformed state /3/.

Let us represent the additional displacements and function p in the form of sums of homogeneous solutions

$$u = U + u^\circ + u_*, \quad v = V + v^\circ + v_*, \quad w = W + w^\circ + w_*, \quad p = P + p^\circ + p_*$$

Using Gauss divergence formula and taking into account that homogeneous solutions satisfy equations of neutral equilibrium /7,1/ and boundary conditions at the plate end-faces, for the variations of functional Π we obtain

$$\delta\Pi = \iint_{o} (\partial_{11}\delta u + \partial_{12}\delta v + \partial_{13}\delta w)\, do \tag{1.7}$$

The procedure with the variation $\delta\Pi$ used below is similar to that described in detail in /4,8/. We denote the boundary values of functions $\Psi(n, s)$, $\partial_1\Psi(n, s)$, $B_m(n, s)$, $C_q(n, s)$, and $C_\nu(n, s)$ at the contour Γ by $\psi(s)$, $\psi_1(s)$, $b_m(s)$, $c_q(s)$, and $c_\nu(s)$, respectively. As in /8/, we introduce the operators K_{ij}, and $S_i(\eta)$ $(i, j = 1, 2)$

$$\Delta\Psi|_{n=0} = K_{11}\psi + K_{12}\psi_1, \quad \partial_1\Delta\Psi|_{n=0} = K_{21}\psi + K_{22}\psi_1 \tag{1.8}$$

$$\partial_1\Phi|_{n=0} = S_1(\eta)\varphi, \quad \partial_1^2\Phi|_{n=0} = S_2(\eta)\varphi \tag{1.9}$$

It is assumed that function $\Phi(n, s)$ satisfies the equation

$$\varepsilon^2\Delta\Phi - \eta^2\Phi = 0 \quad (\operatorname{Re}\eta \neq 0)$$

and $\varphi(s)$ represents the boundary value of $\Phi(n, s)$ on Γ. As in /8/ the asterisk indicates conjugate Lagrange operators (e.g., K_{11}^*).

In the variational equation (1.7) $\delta\psi$, $\delta\psi_1$, δb_m, δc_q, δc_ν are taken as the independent variations. We carry out in (1.7) integration with respect to ζ and, using (1.8) eliminate the dependent variations from the first of formulas (1.9). Then, integrating, where necessary by parts, and using conjugate operators, collect the coefficients at independent variations and equate them to zero. As the result, we obtain for the determination of boundary values of functions Ψ, B_m, C_q, C_ν a homogeneous system of $2 + 3\cdot\infty$ operator equations

$$Q_2 + K_{12}^*(Q_6 + \partial_2 Q_3) + K_{22}^* Q_4 = 0 \tag{1.10}$$

$$\begin{aligned}
&\partial_2 Q_1 + Q_5 + K_{11}^*(Q_6 + \partial_2 Q_3) + K_{21}^* Q_4 = 0\\
&\partial_2 Q_{1k} + S_1^*(\sigma_k) Q_{2k} = 0 \quad (k = 1, 2, 3, \ldots)\\
&\varepsilon^2\partial_2 Q_{1p} + \varepsilon^2 S_1^*(\alpha_p) Q_{2p} + \varepsilon^2 Q_{3p} = 0 \quad (p = 1, 2, 3, \ldots)
\end{aligned}$$

$$\varepsilon^8\partial_2 Q_{1\mu} + \varepsilon^8 S_1^*(\alpha_\mu) Q_{2\mu} + \varepsilon^8 Q_{3\mu} = 0 \quad (\mu = 1, 2, 3, \ldots) \tag{1.11}$$

where

$$\begin{aligned}
Q_1 &= \varepsilon^2 d_1\Delta_{11}\Psi - \varepsilon^3 I_{1m} d_2 B_m + \varepsilon^2 J_{1\varkappa} d_1 C_\varkappa\\
Q_2 &= \varepsilon^2\Delta_{12}\Psi + \varepsilon^3\omega I_{1m} d_1 B_m + (\varepsilon^2 J_{1\varkappa}\partial_1^2 + J_{2\varkappa}) C_\varkappa\\
Q_3 &= \varepsilon^4 d_1\Delta_{21}\Psi - \varepsilon^5 I_{2m} d_2 B_m + \varepsilon^4 J_{3\varkappa} d_1 C_\varkappa\\
Q_4 &= \varepsilon^4\Delta_{22}\Psi + \varepsilon^5\omega I_{2m} d_1 B_m + \varepsilon^2(\varepsilon^2 J_{3\varkappa}\partial_1^2 + J_{4\varkappa}) C_\varkappa\\
Q_5 &= -\partial_1(2\beta^{-1} + \varepsilon^2 R_5\Delta)\Psi + \varepsilon I_{3m}\partial_2 B_m - J_{5\varkappa}\partial_1 C_\varkappa\\
Q_6 &= \varepsilon^2\partial_1(R_6 + \varepsilon^2 R_7\Delta)\Psi - \varepsilon^3 I_{4m}\partial_2 B_m + \varepsilon^2 J_{6\varkappa}\partial_1 C_\varkappa\\
Q_{1k} &= \varepsilon^3\Delta_{1k}\Psi + \varepsilon^4\omega i_{km} d_1 B_m + \varepsilon(\varepsilon^2 j_{k\varkappa}\partial_1^2 + k_{k\varkappa}) C_\varkappa\\
Q_{2k} &= \varepsilon^3\Delta_{2k}\Psi + \varepsilon^4 i_{km} d_2 B_m - \varepsilon^3 j_{k\varkappa} d_1 C_\varkappa\\
Q_{1p} &= \varepsilon^2\Delta_{1p}\Psi - \varepsilon^3\omega^{-1} j_{mp} d_2 B_m + \varepsilon^2 l_{p\varkappa} d_1 C_\varkappa\\
Q_{2p} &= \varepsilon^2\Delta_{2p}\Psi + \varepsilon^3 j_{mp} d_1 B_m + (\varepsilon^2 l_{p\varkappa}\partial_1^2 + m_{p\varkappa}) C_\varkappa\\
Q_{3p} &= -\partial_1(\beta^{-1} J_{8p} + \varepsilon^2 J_{9p}\Delta)\Psi + \varepsilon n_{pm}\partial_2 B_m - r_{p\varkappa}\partial_1 C_\varkappa\\
\Delta_{11} &= R_0 + \varepsilon^2 R_1\Delta, \quad \Delta_{12} = \partial_1^2\Delta_{11} + R_2\Delta\\
\Delta_{21} &= R_1 + \varepsilon^2 R_3\Delta, \quad \Delta_{22} = \partial_1^2\Delta_{21} + R_4\Delta\\
\Delta_{1k} &= \partial_1^2\Delta_{3k} + I_{5k}\Delta, \quad \Delta_{2k} = -d_1\Delta_{3k}, \quad \Delta_{3k} = \omega(I_{1k} + \varepsilon^2 I_{2k}\Delta)\\
\Delta_{1p} &= d_1\Delta_{3p}, \quad \Delta_{2p} = \partial_1^2\Delta_{3p} + J_{7p}\Delta, \quad \Delta_{3p} = J_{1p} + \varepsilon^2 J_{3p}\Delta\\
d_1 &= \rho\partial_2 - \partial_1\partial_2, \quad d_2 = \partial_1^2 - \gamma^2\partial_2^2 - \gamma^2\rho\partial_1\\
R_0 &= \omega\gamma^2\langle\zeta, \zeta\rangle, \quad R_1 = \omega\gamma\langle\zeta, A\rangle, \quad R_2 = \gamma\langle\zeta, C\rangle\\
R_3 &= \omega\langle A, A\rangle, \quad R_4 = \langle A, C\rangle, \quad R_5 = \langle 1, \gamma A' + B\rangle\\
R_6 &= \beta^{-1}\langle 1, B\rangle, \quad R_7 = \langle B, \gamma A' + B\rangle, \quad I_{1k} = \gamma\langle\zeta, F_k\rangle\\
I_{2k} &= \langle A, F_k\rangle, \quad I_{3k} = \gamma\langle 1, F_k'\rangle, \quad I_{4k} = \gamma\langle B, F_k'\rangle\\
I_{5k} &= \langle F_k, C\rangle, \quad i_{km} = \langle F_k, F_m\rangle, \quad j_{kp} = \omega\langle F_k, T_p\rangle\\
k_{kp} &= \langle F_k, N_p\rangle, \quad J_{1p} = \omega\gamma\langle\zeta, T_p\rangle, \quad J_{2p} = \gamma\langle\zeta, N_p\rangle\\
J_{3p} &= \omega\langle A, T_p\rangle, \quad J_{4p} = \langle A, N_p\rangle, \quad J_{5p} = \langle 1, \gamma T_p' - M_p\rangle\\
J_{6p} &= \langle B, \gamma T_p' - M_p\rangle, \quad J_{7p} = \langle T_p, C\rangle, \quad J_{8p} = \langle 1, M_p\rangle\\
J_{9p} &= \langle M_p, \gamma A' + B\rangle, \quad n_{pk} = \gamma\langle M_p, F_k'\rangle, \quad l_{pq} = \omega\langle T_p, T_q\rangle\\
m_{pq} &= \langle T_p, N_q\rangle, \quad r_{pq} = \langle M_p, \gamma T_q' - M_q\rangle
\end{aligned}$$

$$\langle g_1, g_2 \rangle \equiv \int_{-1}^{+1} g_1(\zeta) g_2(\zeta)\, d\zeta$$

Summation from 1 to ∞ is implied by recurring indices, and subscript $\varkappa$ indicates summation with respect to q and ν. For example, $l_{p\varkappa}C_\varkappa \equiv l_{pq}C_q + l_{p\nu}C_\nu$. The prime denotes differentiation with respect to ζ. Quantities $Q_{1\mu}, Q_{2\mu}, Q_{3\mu}$ are obtained, respectively, from Q_{1p}, Q_{2p}, Q_{3p} by the substitution $p \sim \mu$, which also applies to integrals with respect to ζ. For example, $J_{1\mu}, n_{\mu k}$ are obtained from J_{1p}, n_{pk} by the substitution of μ for p; $r_{p\nu}$ is obtained from r_{pq} by substituting ν for q; $l_{\mu\nu}$ from l_{pq} by substituting μ for p and ν for q, and so on. Explicit expressions of integrals are not presented because of their unwieldiness. We only point out that $J_{5p} = 0 = J_{5\mu}$ $(p, \mu = 1, 2, 3, \ldots)$, $i_{km} = 0$ when $k \neq m$.

System (1.10), (1.11) depends on the parameter of initial deformation γ explicitly and implicitly in terms of roots of the characteristic equation. As $\varepsilon \to 0$, it can be considered as a system of integrodifferential equations /4/. A direct inspection shows that when the initial deformation is removed, i.e. when $\gamma \to 1$, system (1.10) (assuming that $C_\nu = 0$) becomes the corresponding system derived in /4/ for the case of an incompressible material. There are, however, two essential differences from /4/, viz: first, the problem considered here is that of stability, i.e. the generalized problem of eigenvalues of system (1.10), (1.11) (as the spectral parameter we have here the quantity γ which is to be determined), and second, we have an additional denumerable set of unknown C_ν and, consequently, an additional system of Eqs. (1.11) that is required for the determination of their boundary values. These aspects complicate the asymptotic analysis of behavior of solutions of system (1.10), (1.11) as $\varepsilon \to 0$. The first because of the complex dependence of this system on γ, and the second owing to the irregular behavior of roots α_ν, as $\gamma \to 1$.

2. According to /1/ the characteristic equation may be presented as follows:

$$\sin \alpha\delta + q(\delta) \sin \alpha (2 + \delta) = 0, \quad \delta = \gamma - 1, \quad q(\delta) = \delta (2 + \delta)^{-1} (\delta^3 - 4\delta - 4)(\delta^3 + 6\delta^2 + 8\delta + 4)^{-1} \tag{2.1}$$

Investigation shows that any positive root of Eq. (2.1) may be represented as

$$\alpha_\mu = \pi\mu\delta^{-1} + x_\mu \quad (\mu = 1, 2, 3, \ldots) \tag{2.2}$$

where x_μ is the solution of equation

$$x = -\delta^{-1} \arcsin \{q \sin [2\pi\mu\delta^{-1} + (2 + \delta) x]\} \tag{2.3}$$

It can be shown that for $\delta > 0$ a unique real solution of Eq. (2.3) exists for any $\mu \geqslant 1$. Since the function $x_\mu(\delta)$ is bounded, it has a singularity at the point $\delta = 0$. Attempts at obtaining an asymptotic formula for $x_\mu(\delta)$ as $\delta \to 0$ proved unsuccessful. Nevertheless, the following expression follows from (2.3):

$$x_\mu(\delta) = \frac{1}{2} a_\mu - \frac{3}{4} \delta a_\mu + \frac{1}{48} \delta^2 a_\mu (30 + a_\mu^2) + \ldots, \quad a_\mu = \sin [2\pi\mu\delta^{-1} + (2 + \delta) x_\mu(\delta)] \tag{2.4}$$

which obviously is an implicit asymptotic expansion of the function $x_\mu(\delta)$. The numerical solution of Eq. (2.3) enables us to obtain approximately any terms of the expansion (2.4). However, for the purpose of this investigation this is insufficient, since in analyzing the asymptotic behavior of solutions of system (1.10), (1.11) for $\varepsilon \to 0$ it is reasonable, taking into account the character of dependence of (1.10) and (1.11) on ε, to seek all of the unknowns in the form

$$\Psi = \Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \ldots, \quad b_m = b_{m0} + \varepsilon b_{m1} + \varepsilon^2 b_{m2} + \ldots \tag{2.5}$$

$$c_q = c_{q0} + \varepsilon c_{q1} + \varepsilon^2 c_{q2} + \ldots, \quad c_\nu = c_{\nu 0} + \varepsilon c_{\nu 1} + \varepsilon^2 c_{\nu 2} + \ldots$$

$$\gamma = 1 + \gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \gamma_3\varepsilon^4 + \ldots \tag{2.6}$$

It follows from (2.6) that it is necessary to obtain expansions in ε of all quantities dependent on γ, in particular of roots $\alpha_0, \alpha_q, \alpha_\nu$. For α_0 and α_q this is trivially solved by the substitution of (2.6) into the perturbation formulas /1/

$$\alpha_0 = \frac{\sqrt{3}}{2} (1 - \gamma^2)^{1/2} \left[1 + \frac{13}{40} (1 - \gamma^2) + O((1 - \gamma^2)^2)\right] \tag{2.7}$$

$$\alpha_q = \alpha_{q0} + \alpha_{q1} (\gamma^2 - 1) + O((\gamma^2 - 1)^2), \quad \sin 2\alpha_{q0} = 2\alpha_{q0}, \quad \alpha_{q1} = -\alpha_{q0} (2 + \operatorname{ctg}^2 \alpha_{q0})/4 \tag{2.8}$$

This proves insufficient for α_ν, since in the substitution in (2.4) of $\gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \ldots$ for δ, all unknown coefficients γ_k $(k \geqslant 1)$ are already present in the first terms of expansion (2.4), which excludes the possibility of their subsequent determination.

This difficulty can be overcome as follows. Investigation shows the estimates

$$x^{\cdot} = O(\delta^{-3}), \quad x^{\cdot\cdot} = O(\delta^{-7}), \quad x^{\cdot\cdot\cdot} = O(\delta^{-11}) \tag{2.9}$$

hold for derivatives $x^{\cdot}(\delta)$, $x^{\cdot\cdot}(\delta)$, $x^{\cdot\cdot\cdot}(\delta)$ calculated on the basis of (2.3) for $\delta \to 0$. Consequently, using the formula of finite increments for $x(\gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \ldots)$ we obtain

$$x(\gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \ldots) = x(\delta_*) + x^{\cdot}(\delta_*)(\gamma_8\varepsilon^9 + \gamma_9\varepsilon^{10}) + {}^1/_2 x^{\cdot\cdot}(\delta_*)\gamma_8{}^2\varepsilon^{18} + O(\varepsilon^5), \quad \delta_* \equiv \gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \ldots + \gamma_7\varepsilon^8 \tag{2.10}$$

The number of terms in (2.10) can be increased by introducing derivatives of higher order and obtaining for them estimates of the type (2.9). Expressing $x^{\cdot}(\delta_*)$ and $x^{\cdot\cdot}(\delta_*)$ in terms of $x(\delta_*)$ and applying formula (2.4) (setting in it $\delta = \delta_*$), from (2.10) we obtain

$$x_\mu(\gamma_1\varepsilon^2 + \gamma_2\varepsilon^3 + \ldots) = {}^1/_2 a_{\mu*} - {}^3/_4\gamma_1\varepsilon^2 a_{\mu*} + (\gamma_8 g_{\mu*} - {}^3/_4\gamma_2 a_{\mu*})\varepsilon^3 + O(\varepsilon^4), \quad a_{\mu*} = \sin\theta_{\mu*}, \; b_{\mu*} = \cos\theta_{\mu*} \tag{2.11}$$

$$\theta_{\mu*} = 2\pi\mu\delta_*{}^{-1} + (2 + \delta_*)x_{\mu*}, \quad x_{\mu*} \equiv x_\mu(\delta_*), \quad g_{\mu*} = \frac{2\pi\mu\varepsilon^2\gamma_1^{-2}q_* b_{\mu*}}{\delta_*\cos x_{\mu*}\delta_* + (2+\delta_*)q_* b_{\mu*}} = O(1), \quad q_* \equiv q(\delta_*)$$

In conformity with (2.2), (2.6), and (2.11) we have

$$\alpha_\mu = \pi\mu\gamma_1{}^{-1}\varepsilon^{-2} - \pi\mu\gamma_2\gamma_1{}^{-2}\varepsilon^{-1} + {}^1/_2 a_{\mu*} + \pi\mu\gamma_1{}^{-3}(\gamma_2{}^2 - \gamma_1\gamma_3) + \pi\mu\gamma_1{}^{-4}(2\gamma_1\gamma_2\gamma_3 - \gamma_2{}^3 - \gamma_1{}^2\gamma_4)\varepsilon + \ldots \tag{2.12}$$

The singularity of formula (2.12) is in that it contains in "bounded" form only the first seven coefficients γ_k. If they are already known, then, solving numerically Eq. (2.3) for $\delta = \delta_*$, it is possible to construct successively all terms of expansion (2.12), as the remaining coefficients are determined.

3. Let us pass to the asymptotic analysis of system (1.10), (1.11) as $\varepsilon \to 0$. We shall seek a solution of the form (2.5), (2.6). Note that unlike in /4/ in our problem operators K_{ij} depend on ε. Taking into account (1.3) and (2.7) it is possible to show the validity of expansion

$$K_{ij} = K_{ij,0} + \varepsilon K_{ij,1} + \varepsilon^2 K_{ij,2} + \ldots$$

The asymptotic expressions for $S_i(\alpha_p)$ and $S_i(\alpha_\mu)$ $(i = 1, 2)$ (and for their conjugate Lagrange operators) is obtained using the data of /4,8/ and formulas (2.6), (2.8), and (2.12). The expansions of $S_i(\sigma_k)$ $(i = 1, 2)$ for $\varepsilon \to 0$ differ from those in /4,8/, since roots σ_k are independent of γ.

We substitute (2.5) and (2.6) into system (1.10), (1.11), and using formulas (2.7), (2.8), and (2.12) and the indicated asymptotic representation of operators, and successively equate to zero the coefficients at $\varepsilon, \varepsilon^2, \varepsilon^3, \ldots$. As the result we obtain in the first approximation with respect to ε the following relations (henceforth the prime denotes a derivative with respect to s):

$$G_1 \sum_{q=1}^{\infty} \varkappa_q c_{q0} = 0 \quad (G_1 \equiv 4\rho e - 8K_{12,0}^*) \tag{3.1}$$

$$G_2 \sum_{q=1}^{\infty} \varkappa_q c_{q0} = 0 \quad (G_2 \equiv 4\partial_2{}^2 - 8K_{11,0}^*) \tag{3.2}$$

$$4\sigma_k b_{k0} + \sum_{q=1}^{\infty} B_{kq} c_{q0}' = 0 \quad (k = 1, 2, 3, \ldots) \tag{3.3}$$

$$\sum_{q=1}^{\infty} A_{pq} c_{q0} = 0 \quad (p = 1, 2, 3, \ldots) \tag{3.4}$$

$$\begin{aligned} &8\pi\mu\gamma_1{}^{-3} c_{\mu0} = 0 \quad (\mu = 1, 2, 3, \ldots) \\ &\varkappa_q = \alpha_{q0}{}^{-1}\sin^2\alpha_{q0}, \quad B_{kq} = 8(-1)^{k+1}\alpha_{q0}(\sigma_k - \alpha_{q0})^{-2}\cos^2\alpha_{q0} \\ &A_{pq} = 8\alpha_{p0}{}^2\alpha_{q0}{}^2(\alpha_{p0} - \alpha_{p0})^{-3}(\cos^2\alpha_{p0} - \cos^2\alpha_{q0}) \quad (p \neq q) \\ &A_{pp} = 4\alpha_{p0}^3\left(1 - \frac{2}{3}\cos^2\alpha_{p0}\right) \end{aligned} \tag{3.5}$$

where e is a unit operator.

The analysis of system (3.4) relative to c_{q0} shows that in the domain of sequences bounded with weight q^2, it has only the trivial solution $c_{q0} = 0$. Obviously Eqs. (3.1) and (3.2) are also satisfied, and from (3.3) we have $b_{k0} = 0$ and from (3.5) $c_{\mu0} = 0$.

Equating to zero the coefficients at ε^2 we obtain (here and subsequently, summation with respect to subscripts q and m is implied, whenever an expression contains not less than two similar subscripts)

$$D_1\Psi_0|_{n=0} + G_1\varkappa_q c_{q1} = 0 \quad (D_1 \equiv {}^4/_3\,(\partial_1^2 + \Delta)) \tag{3.6}$$

$$D_2\Psi_0|_{n=0} + G_2\varkappa_q c_{q1} = 0 \quad (D_2 \equiv -4\gamma_1\partial_1 - {}^8/_3\partial_1\Delta + {}^4/_3\,\partial_2 d_1) \tag{3.7}$$

$$4\sigma_k b_{k1} + 8\,(-1)^k\sigma_k^{-2} d_1\Psi_0|_{n=0} + B_{kq}c_{q1}' = 0 \quad (k = 1, 2, 3, \ldots) \tag{3.8}$$

$$A_{pq}c_{q1} = 0 \quad (p = 1, 2, 3, \ldots) \tag{3.9}$$

$$8\pi\mu\gamma_1^{-3}c_{\mu 1} = 0 \quad (\mu = 1, 2, 3, \ldots) \tag{3.10}$$

From (3.9) we again have $c_{q1} = 0$, and from (3.6) and (3.7) the boundary conditions for Ψ_0

$$D_1\,\Psi_0\,|_{n=0} = 0, \quad D_2\Psi_0\,|_{n=0} = 0 \tag{3.11}$$

The equation for Ψ_0 is obtained from (1.3) using (2.7), and is of the form

$$\Delta^2\Psi_0 + {}^3/_2\,\gamma_1\Delta\Psi_0 = 0 \tag{3.12}$$

It can be shown that (3.12) is the same as the Saint Venant equation of the classical theory of plates stability /6/ based on Kirchhoff's hypotheses, and the boundary conditions (3.11) define in the order of succession the equality to zero of the bending moment and the sum of the shear force and of the torque derivative with respect to s (Ψ_0 coincides within terms of order ε with the buckling of the middle plane). This shows that the classical theory makes possible the correct determination of the principal term of the asymptotic expansion of critical load for a plate with a free edge. With problem (3.11), (3.12) solved for Ψ_0, it is possible to obtain from (3.8) b_{k1}. From (3.10) we again have $c_{\mu 1} = 0$.

In the third approximation with respect to ε we have

$$D_1\,\Psi_1\,|_{n=0} + 8\,(-1)^m\,\sigma_m^{-2}b_{m1}' + G_1\,\varkappa_q c_{q2} = 0 \tag{3.13}$$

$$D_2\,\Psi_1\,|_{n=0} - 4\,\gamma_2\,\partial_1\,\Psi_0\,|_{n=0} + 8\,(-1)^{m+1}\,\sigma_m^{-2}\,(\rho\, b_{m1})' + G_2\varkappa_q c_{q2} = 0 \tag{3.14}$$

$$4\sigma_k b_{k2} - 10\rho b_{k1} + 4\,(-1)^k\,\sigma_k^{-3}\,(2\sigma_k d_1\Psi_1 - \rho d_1\Psi_0)_{n=0} + B_{kq}c_{q2}' = 0 \quad (k = 1, 2, 3, \ldots) \tag{3.15}$$

$$A_{pq}c_{q2} = 0 \quad (p = 1, 2, 3, \ldots) \tag{3.16}$$

$$8\,\pi\mu\gamma_1^{-3}\,c_{\mu 2} = 0 \quad (\mu = 1, 2, 3, \ldots) \tag{3.17}$$

From (3.16) and (3.17) we again have $c_{q2} = 0$, and $c_{\mu 2} = 0$, and from (3.13) and (3.14) the boundary conditions for Ψ_1

$$D_1\Psi_1\,|_{n=0} + 8\,(-1)^m\,\sigma_m^{-2}\,b_{m1}' = 0, \quad D_2\Psi_1\,|_{n=0} - 4\gamma_2\partial_1\Psi_0\,|_{n=0} + 8\,(-1)^{m+1}\,\sigma_m^{-2}\,(\rho b_{m1})' = 0 \tag{3.18}$$

As implied by (1.3), function Ψ_1 satisfies the equation

$$\Delta^2\Psi_1 + {}^3/_2\,\gamma_1\Delta\Psi_1 = -{}^3/_2\,\gamma_2\Delta\Psi_0 \tag{3.19}$$

The solution of the boundary value problem (3.18), (3.19) provides the first corrections to the classical theory: γ_2 and Ψ_1. It is, then, possible to obtain b_{k2} using (3.15).

Without writing down the fourth approximation, we would point out that it generally yields $c_{q3} \neq 0$, while as previously $c_{\mu 3} = 0$. Investigation of subsequent approximations shows that $c_{\mu 4} = c_{\mu 5} = \ldots = c_{\mu 10} = 0$, and that $c_{\mu 11}$ is generally nonzero. Hence it is possible to assert that at least 11 coefficients $\gamma_1, \gamma_2, \ldots, \gamma_{11}$ are independent of potential solutions which are irregular as $\gamma \to 1$. Which means that when the latter are disregarded, the relative error of critical load determination is of an order not lower than ε^{11}. It is also possible to show that the relative error of determination of displacements is of an order not lower than ε^8, and that of additional stresses is of an order of ε^5.

Thus the potential solutions that are irregular as $\gamma \to 1$ have in the case of thin plates a very small effect on the critical force magnitude and on the stress-strain state of the plate. This could have been foreseen based on mechanical considerations, since the equations derived above for γ close to unity strongly oscillates across the thickness, and that, consequently, the buckling of thin plates corresponding to them should be expected to be insignificant.

The above analysis leads to the following conclusions:

1) the principal term of the critical load asymptotics, as $\varepsilon \to 0$ is only determined by the penetrating solution;

2) the first two corrections γ_2 and γ_3 depend in the classical theory on the vortex and, also, on the penetrating solution;

3) the potential solutions that correspond to the regular part of the spectrum affect only the third (and subsequent) correction;

4) in the case of a thin plate the dependence of the critical force on potential solutions that are irregular as $\gamma \to 1$ is very weak, appearing from the eleventh (and possibly higher) correction.

Note that after the determination of $\gamma_1, \gamma_2, \ldots, \gamma_7$ it is possible to numerically determine $x_{\mu *} = x_\mu (\gamma_1 \varepsilon^2 + \ldots + \gamma_7 \varepsilon^8)$, which enables us to obtain asymptotic expansions of all quantities associated with α_μ. This shows that the proposed asymptotic process is constructive. It is, at the same time, difficult to be certain, as in /4/, of its unbounded continuation. Because of the nonuniformity of perturbations of the regular part of potential solutions in the neighborhood of $\gamma = 1$, its termination at some stage cannot be excluded. A similar situation occurs in much simpler problems on eigenvalues /9/.

4. In the case of axisymmetric buckling of a circular plate, we obtain for γ_k the following expressions:

$$\gamma_1 = 2x / 3, \quad \gamma_2 = 0, \quad \gamma_3 = 2\gamma_1^2 (8x - 1) / (5y), \quad \gamma_4 = -32 s_* x^3 / (9y)$$

$$\gamma_5 = \gamma_1^3 (5760x^3 - 10288x^2 + 7392x - 3027) / (140y^3)$$

$$s_* = 0.0762, \quad y = 4x - 3, \quad 2\sqrt{x}\, J_0(\sqrt{x}) - J_1(\sqrt{x}) = 0$$

where J_0 and J_1 are Bessel functions of the first kind. The expression for s_* is obtained by solving the infinite system of linear algebraic equations c_{q3} using the method of truncation with retention of 30 roots α_{q0}. Note that the system in c_{q4} is solved analytically.

From (1.2) and (2.6) we obtain the following expressions for the critical force:

$$t_* / G = 2\gamma_1 \varepsilon^2 + (2\gamma_3 - {}^1/_3\gamma_1^2)\, \varepsilon^4 + 2\gamma_4 \varepsilon^5 + (2\gamma_5 + {}^4/_9\gamma_1^3 - {}^2/_3\gamma_1\gamma_3)\, \varepsilon^6 + \ldots \tag{4.1}$$

where the first term corresponds to the applied theory based on Kirchhoff's hypotheses /6/. It can be shown that $2\gamma_3 - \gamma_1^2 / 3 > 0$. This shows that the classical theory yields an inadequate value for the critical force, with the relative error of the order of ε^2.

For the critical force we obtain from (4.1)

$$t_{*1} / G = 6.254\varepsilon^2 + 14.871\varepsilon^4 - 3.552\varepsilon^5 + 39.273\varepsilon^6 + \ldots \tag{4.2}$$

The magnitude of coefficients in (4.2) makes it possible to expect this asymptotic formula to be valid also for fairly thick plates for which ε reaches the values 0.3—0.4.

The author thanks L. M. Zubov for his interest in this work.

REFERENCES

1. ZUBOV L.M. and RUDEV A.N., Homogeneous solutions for a prestressed elastic plate. PMM, Vol.42, No.5, 1978.

2. LUR'E, A.I., The Theory of Elasticity. Moscow, "Nauka", 1970.

3. ZUBOV L.M., Variational principles of the nonlinear theory of elasticity. Case of superposition of a small deformation on a finite deformation. PMM, Vol.35, No.5, 1971.

4. AKSENTIAN O.K. and VOROVICH I.I., The state of stress in a thin plate. PMM, Vol.27, No.6, 1963.

5. LUR'E A.I., Three-dimensional Problems of the Theory of Elasticity. Moscow, Gostekhizdat, 1955.

6. TIMOSHENKO S.I., Stability of Elastic Systems. Moscow, Gostekhizdat, 1955.

7. ZUBOV L.M., Buckling of plates made of a neo-Hookean material in the case of affine initial deformations. PMM, Vol.34, No.4, 1970.

8. VOROVICH I.I., Some results and problems of the asymptotic theory of plates and shells. Trans. 1-st All-Union School on the Theory and Numerical Methods of Calculation of Shells and Thin Plates, Gegechkori, 1974. Izd. Tbilisi Univ., 1975.

9. KATO T. Perturbation Theory for Linear Operators. Berlin, N.Y., Springer Verlag, 1966.

Translated by J.J.D.